

FFIE/711/116

Approved
Kjeller 6 November 1997

Paul Narum

Director of Research

**WAVELETS FOR DETECTION OF
TRANSIENT SIGNALS**

BREMER Amund Solvi

FFI/RAPPORT-97/05153

FORSVARETS FORSKNING SINSTITUTT
Norwegian Defence Research Establishment
P O Box 25, N-2007 Kjeller, Norway

FORSVARETS FORSKNING SINSTITUTT (FFI)
Norwegian Defence Research Establishment

P O BOX 25
N-2007 KJELLER, NORWAY

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE
(when data entered)

REPORT DOCUMENTATION PAGE

1) PUBL/REPORT NUMBER FFI/RAPPORT-97/05153 1a) PROJECT REFERENCE FFIE/711/116	2) SECURITY CLASSIFICATION UNCLASSIFIED 2a) DECLASSIFICATION/DOWNGRADING SCHEDULE	3) NUMBER OF PAGES 44		
4) TITLE WAVELETS FOR DETECTION OF TRANSIENT SIGNALS				
5) NAMES OF AUTHOR(S) IN FULL (surname first) BREMER Amund Solvi				
6) DISTRIBUTION STATEMENT Approved for public release. Distribution unlimited. (Offentlig tilgjengelig)				
7) INDEXING TERMS IN ENGLISH: <table style="width: 100%; border: none;"> <tr> <td style="width: 50%; vertical-align: top;"> a) <u>Wavelets</u> b) <u>Matched filters</u> c) <u>Detectors</u> d) <u>Transient</u> e) <u>Matlab</u> </td> <td style="width: 50%; vertical-align: top;"> IN NORWEGIAN: a) <u>Wavelets</u> b) <u>Matched filtre</u> c) <u>Detektorer</u> d) <u>Transienter</u> e) <u>Matlab</u> </td> </tr> </table>			a) <u>Wavelets</u> b) <u>Matched filters</u> c) <u>Detectors</u> d) <u>Transient</u> e) <u>Matlab</u>	IN NORWEGIAN: a) <u>Wavelets</u> b) <u>Matched filtre</u> c) <u>Detektorer</u> d) <u>Transienter</u> e) <u>Matlab</u>
a) <u>Wavelets</u> b) <u>Matched filters</u> c) <u>Detectors</u> d) <u>Transient</u> e) <u>Matlab</u>	IN NORWEGIAN: a) <u>Wavelets</u> b) <u>Matched filtre</u> c) <u>Detektorer</u> d) <u>Transienter</u> e) <u>Matlab</u>			
THESAURUS REFERENCE:				
8) ABSTRACT A review of basic wavelet theory is given, with particular emphasis on applications in signal processing. Several methods for detecting transient signals in a noisy background are presented. It is shown how the classical matched filter detector can be efficiently computed in the wavelet domain. The wavelet based matched filter is shown to have greater flexibility and superior performance compared to its classical time domain counterpart. The wavelet based matched filter algorithm is implemented in Matlab.				
9) DATE 5 November 1997	AUTHORIZED BY This page only Paul Narum <i>PW</i>	POSITION Director of Research		

ISBN 82-464-0214-5

UNCLASSIFIED

CONTENTS

	Page	
1	INTRODUCTION	5
2	TIME-FREQUENCY SIGNAL REPRESENTATIONS	6
2.1	The short-time Fourier transform	6
2.2	Frames for general signal transforms	8
2.3	Wavelet transforms	10
2.4	Multiresolution analysis	12
2.5	Wavelet construction by sub-band filtering	13
2.6	Construction of the orthonormal wavelet bases	15
3	SPECIAL TOPICS AND FURTHER GENERALISATIONS	21
3.1	The fast wavelet transform	21
3.2	Time invariance	22
3.3	Wavelet packets	23
4	TRANSIENT DETECTION AND CHARACTERISATION	25
4.1	Preliminary considerations	25
4.2	Matched filter	26
4.2.1	Time domain	26
4.2.2	Transform domain	27
4.3	Non-parametric detectors	28
4.3.1	Energy detector	28
4.3.2	Maximum transform coefficient detector	29
5	PRACTICAL EXAMPLES	30
5.1	Matched filter	30
6	SUMMARY AND FURTHER WORK	32
6.1	Summary	32
6.2	Further work	32
	List of symbols	34
	References	35
APPENDIX		
A	MEASURE THEORY	36

B	COMPUTER PROGRAMS	38
B.1	Short documentation	38
B.2	Program listings	39
	Distribution list	44

WAVELETS FOR DETECTION OF TRANSIENT SIGNALS

1 INTRODUCTION

This report documents the work done by the author in connection with the fourth year subject “*Prosjekt 2*” at the Norwegian University of Science and Technology, Department of Mathematical Sciences. The work has been carried out at The Norwegian Defence Research Establishment (FFI) as a part of the development project CHES (Composite Hull Embedded Sensor System), which is a joint undertaking between FFI and the Naval Research Laboratory, Washington DC.

The goal of the CHES project is to develop a real-time system to monitor and analyse the load on the hull of a high speed Surface Effect Ship (SES) which is currently being built at Kværner Mandal, Norway.

This report focuses on the problem of detecting signals arising from *slamming*, heavy impacts between waves and the hull of the ship. These signals tend to be oscillatory and of relative short duration, masked by non-white noise. There are several reasons for the interest in these transient signals; firstly, one would like to know during operation if ship is going so fast that the plates in the hull risk being subjected to unsustainable loads; secondly, there is reason to believe that changes over time in the vibrational frequency and its higher harmonics might indicate structural damages.

Two other reports on wavelets and wavelet-based detection schemes related to the CHES project have already been published [1], [2]. This report should be seen in conjunction with them, so we do not needlessly repeat the discussion in those. Instead, we try to present a coherent and mathematical overview of wavelet theory and signal transforms. In particular, we have tried to develop a common notation.

First, we briefly review selected parts of wavelet theory in Sections 2 and 3. Several methods for detecting and characterising transient signals are suggested in Section 4. These methods are implemented in MATLAB and tested on experimental data. Some results of these tests are presented in Section 5. Finally, we point out some areas into which more effort should be put.

2 TIME-FREQUENCY SIGNAL REPRESENTATIONS

Frequency analysis of signals and functions is a method which is indispensable in many parts of science and engineering. Fourier analysis is the most well known of these techniques, where a function is analysed as a linear combination of sinusoids. The relative magnitude of the coefficients of such a linear combination gives information on the importance of different frequency components of the function.

However, as a sinusoid with given phase and frequency does not change character over time, such a linear combination weights frequency components equally at all times. This is a problem if we are interested in knowing the frequency components of phenomena which are transient or non-stationary. What we then need is a transform which can characterise a function by its frequency contents at different times. A tutorial and an extensive literature survey on several such time-frequency signal transforms (STFT) can be found in [3]. We will restrict ourselves to review the short-time Fourier transform and the wavelet transform.

2.1 The short-time Fourier transform

The short-time Fourier transform (STFT) of a function $f(\cdot) \in L^2(\mathbb{R})$ is defined as

$$(T^{STF} f)(\omega, t) \equiv \tilde{f} = \int_{-\infty}^{\infty} d\omega f(u) e^{-2\pi i \omega u} \bar{w}(u - t), \quad w \in L^2(\mathbb{R}) \quad (2.1)$$

where $w(\cdot)$ is a *window function*. The reason for the complex conjugate appearing in (2.1) will soon become apparent. We recognise this as the ordinary Fourier transform of f multiplied with a shifted version of \bar{w} . Thus, if $w(u)$ is located around $u = 0$, then, for each t , $(T^{STF} f)(\omega, t)$ will be the (approximate) ‘instantaneous’ frequency distribution of f near that time.

What we really mean by instantaneous frequency is a little unclear, however. We know that precise simultaneous measurements of time and frequency are mutually incompatible; the uncertainty principle states that this as an intrinsic property of functions. We will confine ourselves to make an intuitive interpretation, and refer to [5] for more details.

Before we proceed to find an inverse of (2.1), we give an example. A discretization of the STFT can be computed by taking the DFT of a signal multiplied with a suitable window function. This is shown in Figure 2.1, where a Gaussian window is used as in [2]. The signal shown is data from a full scale measurement of the strain on the hull of a ship. We clearly see two strong transients, and from the bottom of the STFT plot we can identify a recurring component every 0.5s, consisting of oscillations around 2–3Hz. We can also identify higher frequency components, especially in the transient around time 2s. In the original computer plots, we see a well defined 20Hz component, and a few much fainter higher frequency components, which might represent higher order vibration modes. This might be difficult to see on print due to poor printing resolution. There are also theoretical justifications for this interpretation; finite element models show that the frequencies around 2.5Hz correspond to beam mode vibrations, whereas

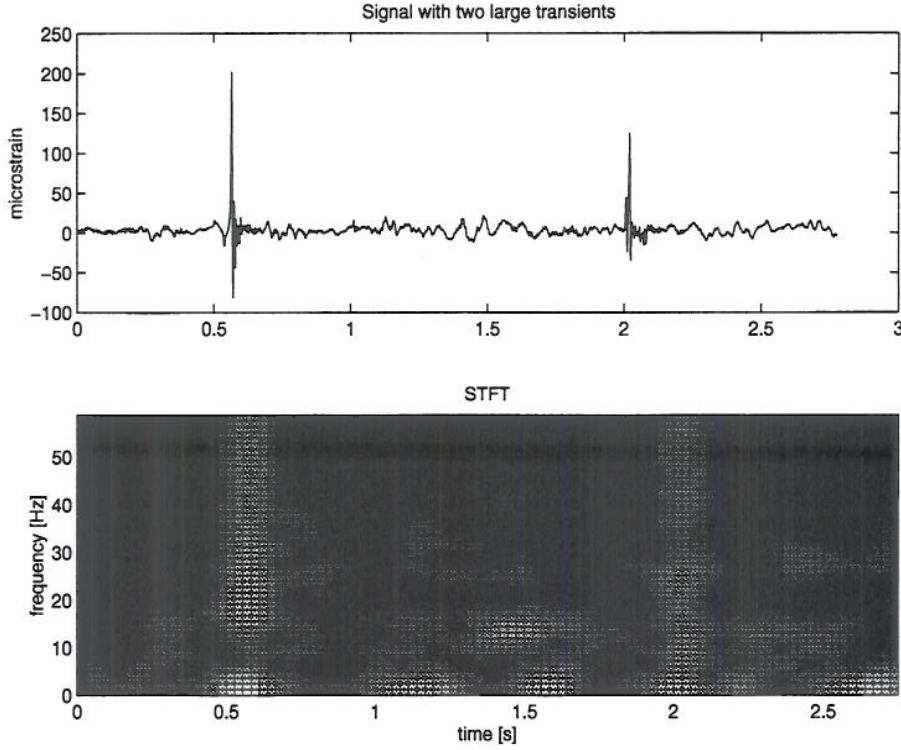


Figure 2.1 A signal with two large transient and its discrete short-time Fourier transform.

the plates in the hull will undergo plate vibrations with a fundamental frequency around 20Hz and super-harmonic components at higher frequencies.

To find an inverse, or *synthesising*, transform we write

$$\tilde{f}(\omega, t) = \mathcal{F}(f(u)\bar{w}(u-t)). \quad (2.2)$$

Applying the inverse Fourier transform, we get

$$f(u)\bar{w}(u-t) = \int_{-\infty}^{\infty} d\omega \tilde{f}(\omega, t) e^{2\pi i \omega u} \quad (2.3)$$

Writing

$$g_{\omega, t}(u) = e^{2\pi i \omega u} w(u-t), \quad (2.4)$$

we get upon multiplying (2.3) by $w(u-t)$, integrating over t and reversing the order of integration

$$f(u)C = \iint d\omega dt \tilde{f}(\omega, t) g_{\omega, u} e^{2\pi i \omega u} \quad (2.5)$$

where $C = \|w\|_{L^2}^2$. We have for convenience suppressed the limits of integration. Using an inner product formalism, we thus have the following formulae for analysis

and synthesis;

$$\tilde{f} = \langle f, g_{\omega,t} \rangle_{L^2(\mathbb{R})}, \quad (2.6)$$

$$f = C^{-1} \langle \tilde{f}, \bar{g}_{\omega,t} \rangle_{L^2(\mathbb{R}^2)}. \quad (2.7)$$

The latter equation can be written

$$f = \langle \tilde{f}, g^{\omega,t} \rangle_{L^2(\mathbb{R}^2)} \quad (2.8)$$

where we have introduced the *reciprocal family* $g^{\omega,t} = \|g\|_{L^2(\mathbb{R})}^{-2} \bar{g}_{\omega,t}$. Substituting the (2.6) into (2.7), we have a *resolution of the identity*,

$$f = \iint d\omega dt \bar{g}^{\omega,t} \langle f, g_{\omega,t} \rangle_{L^2(\mathbb{R})} \quad (2.9)$$

In operator terms, this is

$$\text{Id} = \iint d\omega dt \bar{g}^{\omega,t} \langle \cdot, g_{\omega,t} \rangle_{L^2(\mathbb{R})}, \quad (2.10)$$

where Id is the identity operator. The above equations are all part of a broader structure. To better understand that theory, we first need to investigate the concept of *frames*.

2.2 Frames for General Signal Transforms

By using the theory of (generalised) frames, we will be able to obtain a general theory for many Fourier and wavelet types of transforms. The advantage of this approach is [6]

...that certain types of theorems (such as reconstruction formulas, consistency conditions, and least-square approximations) do not have to be proved again and again in different settings; instead they can be proven once and for all in the setting of generalised frames. Since the field of wavelets is so new, it is important to keep a broad spectrum of options open concerning its possible course of development.

Frames are very reminiscent of the familiar concept *bases*. In fact, any basis is a frame, but the opposite need not be true. In a space with a finite dimension, say N , a basis is a set of N linearly independent vectors. Since these vectors are linearly independent, they *span* the space. A frame in the same space would also be a set of vectors spanning the space, but they may be *linearly dependent*. Hence, a frame can contain more than N vectors, and may therefore be *redundant*. In infinite dimensional spaces, the situation is similar.

In order to be as general as possible we shall make slight use of measure theory; we have provided a short review in Appendix A for readers unfamiliar with this concept.

Definition 1 (Frames) Let H be a Hilbert Space and let M be a measure space with measure μ . A frame in H indexed by M is a family of vectors $\{e_m \in H : m \in M\}$ such that

1. For every $f \in H$, the function $\tilde{f} : M \rightarrow \mathbb{C}$ defined by

$$\tilde{f} = \langle f, e_m \rangle \quad (2.11)$$

is measurable.

2. There is a pair of constants $0 < A \leq B < \infty$ such that for every $f \in H$,

$$A\|f\|_H^2 \leq \|\tilde{f}\|_{L^2(\mu)}^2 \leq B\|f\|_H^2 \quad (2.12)$$

The vectors $\{e_m\}$ are called *frame vectors*, A and B are called the *frame bounds*, and (2.12) is called the *frame condition*. From the frame condition, we see that $\tilde{f} \in L^2(\mu)$ since A and B are finite. Therefore (2.11) can be written $(Tf)(m) = \langle f, e \rangle_H$, where we have introduced the *frame operator* $T : H \rightarrow L^2(\mu)$. We call $\tilde{f}(m) = (Tf)(m)$ the transform of f with respect to the frame. Sometimes we will write T_ϕ if we want to emphasise that we are using the transform with respect to some function ϕ as in $(T_\phi f)(m)$. For example, with this notation the Fourier transform of a function f would be written $(T_{e^{2\pi i \omega t}} f)(\omega)$.

To reconstruct f from its transform, we need the left inverse of T . We will call this operator the *synthesising operator* S . We first note that since T is an operator on Hilbert spaces, it has a unique adjoint, T^* , [7].

We introduce G , the metric operator

$$G \equiv T^*T : H \rightarrow H, \quad (2.13)$$

and note that G is self-adjoint, so $G = G^*$. Since now $\|Tf\|_{L^2(\mu)} = \langle Gf, f \rangle_H$, we can write the frame condition (2.12) as

$$A\langle f, f \rangle_H \leq \langle Gf, f \rangle_H \leq B\langle f, f \rangle_H. \quad (2.14)$$

By subtracting the right and left hand sides of this inequality from the middle part, we see that $\|G\| \leq B$, and $\|G\| \geq A$ almost everywhere. Therefore, G is bounded, and, more importantly, G^{-1} exists and is bounded by A^{-1} .

Now it is easy to show that S defined by $S = G^{-1}T^*$ is a left inverse of T . Indeed, if $Tf = g$, then

$$Sg = G^{-1}T^*(Tf) = G^{-1}Gf = f. \quad (2.15)$$

However, one should note that S need not be a two-sided inverse, that is, $TS \neq \text{Id}$ in general. We recognise S as a pseudo-inverse-like operator; if $g(m)$ is an arbitrary signal in the transform space, $f = Sg$ minimises the discrepancy $\|Tf - g\|^2$.

It can be shown that the action of S is given by

$$Sg = \langle g, e^m \rangle_{L^2(\mu)}, \quad (2.16)$$

where $e^m = G^{-1}\bar{e}_m$ is the *reciprocal family* to e_m . The use of complex conjugation in this definition is non-standard, but allows us to write the reconstruction formula as an inner product.

We call a frame *tight* when $A = B$. The frame condition becomes

$$A\langle f, f \rangle_H = \langle Gf, f \rangle_H = B\langle f, f \rangle_H. \quad (2.17)$$

In other words, $\langle Gf, f \rangle = A\langle f, f \rangle$ almost everywhere, so

$$G = A \text{Id} \quad \text{and} \quad G^{-1} = A^{-1} \text{Id}. \quad (2.18)$$

The reciprocal family is then given simply as

$$e^m = A^{-1}\bar{e}_m \quad (2.19)$$

Thus, a general transform pair in a tight frame can be written

$$(Tf)(m) = \tilde{f}(m) = \langle f, e_m \rangle_H \quad (2.20)$$

$$(S\tilde{f})(t) = f(t) = \langle \tilde{f}, e^m \rangle_{L^2(\mu)} \quad (2.21)$$

This is to be compared to (2.6 and (2.7). We will now discuss one very special class of frame vectors, the *wavelets* $\{\psi_{a,b}\}$.

2.3 Wavelet transforms

In the short-time Fourier analysis, we have seen that a function is decomposed by means of functions $g_{\omega,t}$ which are generated by translation of a window function, and multiplied by an oscillating sinusoid, (2.4). In wavelet analysis, we likewise decompose the function by means of basis given by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right). \quad (2.22)$$

Here, the different basis functions are generated from a *mother wavelet* by means of scaling by a and translation by b . The normalisation constant ensures that all $\psi_{a,b}$ has the same L^2 -norm. The wavelets should be well localised in both time and frequency in order to separate different frequency components occurring at different times, see

[8]. Under certain conditions, the wavelets will form a tight frame, as in the previous section. The continuous wavelet transform and its inverse are [9]

$$(T^{CW} f)(a, b) = \tilde{f}(a, b) = \langle f, \psi_{a,b}(t) \rangle_{L^2(\mathbb{R})} = \int dt f(t) \overline{\psi_{a,b}(t)} \quad (2.23)$$

$$(S^{CW} \tilde{f})(t) = f(t) = \langle \tilde{f}, \psi^{a,b} \rangle_{L^2(\mu)} = C_\psi^{-1} \iint \frac{da db}{a^2} \tilde{f}(a, b) \psi_{a,b}(t). \quad (2.24)$$

The measure μ is given by $d\mu = da db/a^2$, and $(a, b) \in \mathbb{R} \times \mathbb{R}$. The frame constant can be calculated explicitly by

$$C_\psi = \int d\omega |\omega|^{-1} \hat{\psi}(\omega), \quad (2.25)$$

which must be finite, according to Definition 1. This condition, called the *admissibility* condition, is usually equivalent with demanding that $\int \psi(t) dt = 0$. We shall not go into further details on sufficient or necessary conditions for the decomposition to be valid. The statement that $\int \psi(t) dt = 0$ means that ψ is in some sense oscillating. We see that for low scales a , the function $\psi_{a,b}$ will be compressed and thus oscillate faster, whereas for high scales it is stretched out and oscillates more slowly. Hence, we can identify high scale with low frequencies, and vice versa.

For practical purposes, there is limited interest in the continuous wavelet transform. The *discrete wavelet transform* is much to prefer, as it is less redundant and there exist efficient algorithms for calculating the inner products (2.23) and (2.24). The DWT follows from the continuous transform by a discretization of the parameters a, b . This is different from the discrete and continuous Fourier transforms, where there in general is no such simple relationship. The discretization scheme usually adopted is $a = a_0^j, b = kb_0 a_0^j$, where $a_0 > 1, b_0 > 0$ and $j, k \in \mathbb{Z}$. This scheme leads to the wavelet basis

$$\psi_{j,k} = a_0^{-j/2} \psi(a_0^{-j} t - kb_0). \quad (2.26)$$

We will consider only $a_0 = 2$, and by a simple rescaling argument, we can without loss of generality assume $b_0 = 1$, see [9], Chapter 1.

We have seen that the short-time Fourier transform and the wavelet transform both fit nicely into the same theoretical framework. As for the dissimilarities, one of the more crucial differences can be seen by comparing the basis functions in (2.4) and (2.22). The basis used in the STFT has constant width at all frequencies, which means that we are using the same time resolution to describe low-frequency and high-frequency phenomena. This is obviously a problem if we need to analyse signals of transient nature with different spectral distributions; if we use a very narrow window to catch the highest frequencies, we are sacrificing good frequency resolution at the lower frequency in the process. Similarly, if we use a broad window, we are in a sense averaging out the highest frequencies. The wavelets are much more flexible; we have identified high frequencies with low scale, and at a low scale the wavelet is narrow, giving good resolution. Similarly, at large scale the wavelet is broad and analyses low frequencies

well. This "zooming" capability of the wavelet transform is one of the reasons why it has met a certain degree of success as a tool for analysing transient phenomena, whether they are found in physics, economics, signal processing or image compression.

2.4 Multiresolution analysis

So far, we have formulated the theory of wavelets in an abstract mathematical setting, and we have made use of Hilbert space formalism. As it stands, our formulation is not direct amenable for algorithmization. We will see that the concept of multiresolution leads to a formulation of the basic theory which lends itself to an easy implementation on a computer. The resulting algorithm will be very fast, allowing us to compute the coefficients in the expansion in $O(N)$ time.

Definition 2 (Multiresolution) *A sequence of subspaces is called a Multiresolution Analysis if there exists a sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces in $L^2(\mathbb{R})$ satisfying*

$$V_j \subset V_{j+1} \tag{2.27}$$

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \tag{2.28}$$

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\} \tag{2.29}$$

$$\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}) \tag{2.30}$$

$$\{\phi(t - k)\}_{k \in \mathbb{Z}} \text{ is an orthonormal basis for } V_0 \tag{2.31}$$

These basis function also have to satisfy some additional requirements if they are to generate a multiresolution analysis; see Chapters 1 and 5 in [9], and also [10]. Furthermore, we may generalise the dilation factor 2 in (2.28) to any rational number, and the $\{\phi(t - k)\}$ may constitute a so called Riesz basis, but this will not be discussed in this report. As for the interpretation of these spaces, we claim that with increasing the spaces contain finer and finer details. See for example Chapter 6 in [12].

Let us introduce the wavelet spaces W_j and finalise the multiresolution formulation for the wavelet transform.

Definition 3 (Wavelet spaces) *Assume that the sequence $\{V_j\}_{j \in \mathbb{Z}}$ constitutes a multiresolution analysis. The wavelet space W_j is then defined as the orthogonal complement of V_j in V_{j-1} ,*

$$V_{j-1} = V_j \oplus W_j. \tag{2.32}$$

Thus, we can say that the wavelet space at level j contains the extra detail required for going from V_j to the higher resolution space V_{j-1} . Defined in this way the different

spaces W_j are orthogonal. If we have a high detail space at level 0, we may decompose it in mutually orthogonal subspaces by iterating (2.32) as follows

$$V_0 = V_J \oplus \left(\bigoplus_{k=1}^J W_k \right). \quad (2.33)$$

Assuming that certain technical conditions are satisfied, we may derive an orthonormal (or Riesz) basis for W_0 . We denote this basis by $\{\psi(t - k)\}_{k \in \mathbb{Z}}$, and we will derive it in Section 2.6. The crucial observation we now make, is that the wavelet spaces, and consequently their bases, inherit the scaling property (2.28) from the scaling spaces. Then an orthogonal basis for W_j is $\{\psi(2^{-j}t - k)\}_k$, or $\{\psi_{j,k}(t)\}_k$ where we recall the notation from (2.26). We similarly define

$$\phi_{j,k}(t) = 2^{-j} \phi(2^{-j}t - k) \quad (2.34)$$

Suppose now that we have a function $f \in V_0$. From (2.33) we can conclude that the family $\{\phi_{J,k}, \psi_{1,k} \dots \psi_{J,k}\}$ constitutes an orthonormal basis and tight frame for V_0 . Since the frame is orthonormal, the reciprocal family is given by the complex conjugate of the original family, as in (2.19). We may write (2.21) as

$$f(t) = \sum_{k=-\infty}^{\infty} c_J[k] \phi_{J,k}(t) + \sum_{j=1}^J \sum_{k=-\infty}^{\infty} d_j[k] \psi_{j,k}(t). \quad (2.35)$$

where

$$c_J[k] = (\mathbb{T}_\phi f)(J, k) = \langle f, \phi_{J,k} \rangle \quad (2.36)$$

$$d_j[k] = (\mathbb{T}_\psi f)(J, k) = \langle f, \psi_{j,k} \rangle \quad j = 1, \dots, J \quad (2.37)$$

The latter pair are (2.20) for our particular choice of basis. These are the wavelet synthesis and analysis formulae. We use the filter engineering convention of indicating the time-like variable k with brackets in anticipation of the next subsection. There we will show how the coefficients can be calculated in an efficient way using sub-band filtering.

2.5 Wavelet transform by sub-band filtering

From the definition of a multiresolution analysis we see that $V_0 \subset V_{-1}$ so $\phi_{-1,k}$ is an orthonormal basis for V_{-1} . It follows that there exists a sequence $h_0[k] \in l^2(\mathbb{Z})$, such that

$$\phi(t) = \sqrt{2} \sum_k h_0[k] \phi(2t - k). \quad (2.38)$$

As we shall see, this equation is fundamental – everything follows from it. Similarly, as $W_0 \subset W_{-1} \subset V_{-1}$, we may express a function in W_0 as a linear combination of functions in V_{-1} ;

$$\psi(t) = \sqrt{2} \sum_k h_1[k] \phi(2t - k) \quad (2.39)$$

The two sequences $h_0[k]$ and $h_1[k]$ are both square sumable, and the two equations are commonly called the *scaling identity* and the *wavelet identity*, respectively. They are not independent; in Section 2.6 we will indirectly show that $h_1[k]$ follows from $h_0[k]$. For explicit formulae, see [9]. To sum up; if we have the sequence $h_0[k]$, we can, in theory, solve (2.38) for $\phi(t)$, and then find $\psi(t)$ from the wavelet identity. In this way, we can find all the expansion functions in (2.35). But there is more; The two sequences $h_0[k]$ and $h_1[k]$ also give us a fast way of obtaining the *inner products* (2.36) and (2.37) we need. Let us state a key theorem, which is the basis for the fast wavelet transform FWT and its inverse.

Theorem 1 *Let $c_j[k]$ and $d_j[k]$ be defined as in (2.36) and 2.37, respectively. Then, for $j = 1, \dots, J$ the following relationships hold.*

$$c_j[k] = \sum_n \overline{h_0[n - 2k]} c_{j-1}[n] \quad (2.40)$$

$$d_j[k] = \sum_n \overline{h_1[n - 2k]} c_{j-1}[n] \quad (2.41)$$

$$c_{j-1}[k] = \sum_n (h_0[n - 2k] c_j[n] + h_1[n - 2k] d_j[n]) \quad (2.42)$$

We see from (2.40) and (2.41) that if we have the fine-scale coefficients $c_0[k]$, we can recursively find the coarser scale coefficients. This is precisely the *analysis step* in a wavelet decomposition.

On the other hand, if we have the coefficients $c_j[k]$ and $d_j[k]$ for $j = 1, \dots, J$ in the wavelet decomposition (2.35), we can recursively recover the finer scale scaling coefficients by means of (2.42). This process is the *synthesis step* in a wavelet reconstruction.

The relatively simple proof of these relationships as they are stated here can be found in [11]. We restrict ourselves to note that the two sequences $h_0[k]$ and $h_1[k]$ are the impulse responses of a high pass and a low pass filter, respectively. In fact, these two filters are *quadrature mirror filters*. Such so-called QMFs have several desirable properties, and there is a comprehensive literature on them in the field of filter theory.

The transform can be represented pictorially as in Figure 2.2. The symbol $\downarrow 2$ means downsampling by two, that is, removing every second element of the input sequence. Here, a capital letter, e.g. A , inside a filter symbol denotes convolution with the corresponding sequence $\{a_n\}_n$. By making this connection between filter theory and multiresolution, we have almost done away with the inner product formalism. Furthermore, we have provided a framework for applying wavelets to many problems in many different parts of signal processing.

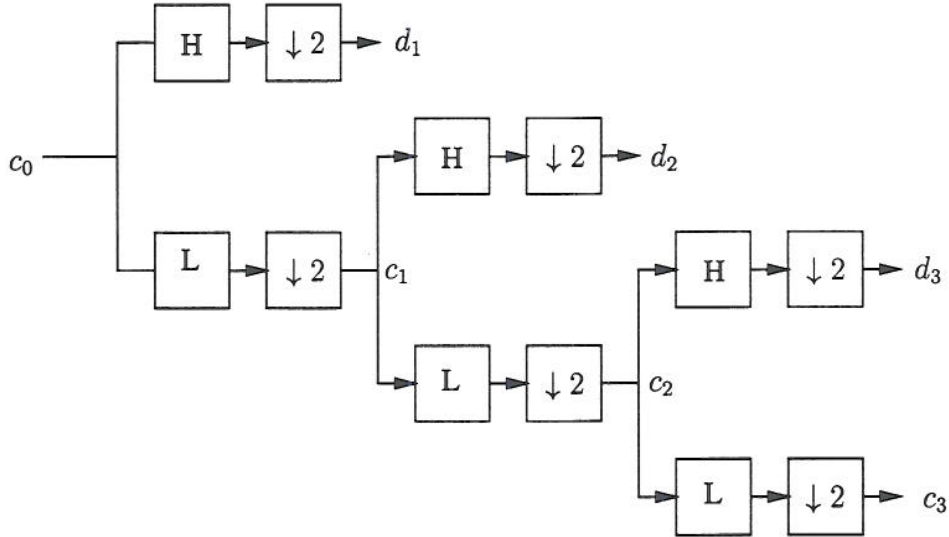


Figure 2.2 Schematical representation of the three stage wavelet decomposition structure.

As one might expect, not just any two filters will give rise to a multiresolution analysis. Details on filtering aspects, necessary and sufficient conditions on the various quantities can be found in [9], [10] and [11] Chapter 5, amongst others.

2.6 Construction of the orthonormal wavelet bases.

In this section, we prove as advertised the following result:

Theorem 2 (Orthonormal wavelet basis) *Let the sequence of subspaces $\{V_j\}$ in $L^2(\mathbb{R})$ constitute a multiresolution analysis, and let $\{\phi_{0,k}\}_{k \in \mathbb{Z}}$ be an orthonormal basis for V_0 . Then we may derive an orthonormal basis $\{\psi_{0,k}\}_{k \in \mathbb{Z}}$ for the space W_0 defined by $V_{-1} = V_0 \oplus W_0$.*

Following [9], we consider an arbitrary function $f \in W_0$. We investigate this function and construct a function ψ (our wavelet) whose translates span W_0 .

Since $V_{-1} = V_0 \oplus W_0$, f is also in V_{-1} , and $f \perp V_0$. We know that $\{\phi_{-1,k}\}_k$ is a basis for V_{-1} , and consequently we can express f as

$$f = \sum_{k \in \mathbb{Z}} f_k \phi_{-1,k}, \quad (2.43)$$

where $f_k = \langle f, \phi_{-1,k} \rangle$. We will need the Fourier transform of f , so let us first find an expression for \hat{f} . From (2.43),

$$\hat{f} = \sum_{k \in \mathbb{Z}} f_k \sqrt{2} \mathcal{F} \phi(2t - k) \quad (2.44)$$

$$= \sqrt{2} \sum_{k \in \mathbb{Z}} f_k \frac{1}{2} e^{2\pi i \omega / 2k} \hat{\phi}(\omega/2) \quad (2.45)$$

$$\equiv m_f(\omega/2) \hat{\phi}(\omega/2), \quad (2.46)$$

where we in the last equality have introduced the function

$$m_f(\omega) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} f_k e^{2\pi i \omega k}. \quad (2.47)$$

This equality holds almost everywhere. For future use, we record the fact that m_f is periodic with period 1, and that $m_f \in L^2([0, 1])$. Much of this section is devoted to finding this m_f .

To do just that, we need the Fourier transform of the scaling function ϕ ; Starting from the scaling identity (2.38), we get

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} h_0[k] e^{-2\pi i \omega k/2} \hat{\phi}(\omega/2) \quad (2.48)$$

$$\equiv m_0(\omega/2) \hat{\phi}(\omega/2), \quad (2.49)$$

where we have introduced the $L^2([0, 1])$ -function $m_0(\omega)$ with period 1 defined by $m_0(\omega) = 1/\sqrt{2} \sum_k h_0[k] e^{2\pi i \omega k}$. We also need another property of ϕ , which follows from the orthonormality of $\phi_{0,0}$ and Parseval's Theorem;

$$\delta_{0,k} = \langle \phi_{0,0}, \phi_{0,k} \rangle_{L^2(\mathbb{R})} \quad \forall k \in \mathbb{Z} \quad (2.50)$$

$$= \langle \mathcal{F}\phi_{0,0}, \mathcal{F}\phi_{0,k} \rangle_{L^2(\mathbb{R})} \quad \text{Parseval's theorem} \quad (2.51)$$

$$= \langle \hat{\phi}, \hat{\phi} e^{-2\pi i \omega k} \rangle_{L^2(\mathbb{R})} \quad (2.52)$$

$$= \int_{-\infty}^{\infty} d\omega |\hat{\phi}|^2 e^{2\pi i \omega k} \quad (2.53)$$

$$= \int_0^1 d\omega \sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega + n)|^2 e^{2\pi i (\omega + n)k} \quad (2.54)$$

$$= \int_0^1 d\omega e^{2\pi i k \omega} \sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega + n)|^2. \quad (2.55)$$

The last equality follows from splitting up the integration domain into intervals of length 1 and noting that $e^{2\pi i n k} \equiv 1$. Equation (2.55) is the inverse Fourier transform of the function

$$\sum_{n \in \mathbb{Z}} |\phi(\omega + n)|^2. \quad (2.56)$$

By taking the Fourier transform on both sides, we obtain

$$\sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega + n)|^2 = 1 \quad \text{a.e.}, \quad (2.57)$$

a result we will need later. The equality holds almost everywhere because the Fourier Transform is an unitary operator on the class of all L^1 -functions differing only on sets

with measure zero. Substituting (2.49) into this equation and splitting the sum into two parts corresponding to odd and even n , we obtain

$$1 = \sum_{n \in \mathbb{Z}} \left| m_0 \left(\frac{\omega + n}{2} \right) \hat{\phi} \left(\frac{\omega + n}{2} \right) \right|^2 \quad (2.58)$$

$$\begin{aligned} &= \sum_{n \in \mathbb{Z}} \left| m_0 \left(\frac{\omega + 2n}{2} \right) \hat{\phi} \left(\frac{\omega + 2n}{2} \right) \right|^2 \\ &\quad + \sum_{n \in \mathbb{Z}} \left| m_0 \left(\frac{\omega + 2n + 1}{2} \right) \hat{\phi} \left(\frac{\omega + 2n + 1}{2} \right) \right|^2 \end{aligned} \quad (2.59)$$

$$\begin{aligned} &= \sum_{n \in \mathbb{Z}} \left| m_0(\omega' + n) \hat{\phi}(\omega' + n) \right|^2, \quad \omega' = \omega/2 \\ &\quad + \sum_{n \in \mathbb{Z}} \left| m_0 \left(\omega' + n + \frac{1}{2} \right) \hat{\phi} \left(\omega' + n + \frac{1}{2} \right) \right|^2 \end{aligned} \quad (2.60)$$

$$= |m_0(\omega')|^2 \cdot 1 + |m_0(\omega' + \frac{1}{2})|^2 \cdot 1. \quad (2.61)$$

In the last equality, we used the periodicity of m_0 and (2.57). Our main result is that

$$|m_0(\omega)|^2 + |m_0(\omega + \frac{1}{2})|^2 = 1. \quad (2.62)$$

Using the fact that f is orthogonal to $V_0 = \text{span}_k \{ \phi_{0,k} \}$ we can now finally investigate f in more detail;

$$0 = \langle f, \phi_{0,k} \rangle \quad \forall k \in \mathbb{Z} \quad (2.63)$$

$$= \langle \mathcal{F}f, \mathcal{F}\phi_{0,k} \rangle \quad \text{Parseval's Theorem} \quad (2.64)$$

$$= \langle \hat{f}, \hat{\phi} e^{-2\pi i \omega k} \rangle \quad (2.65)$$

$$= \int_{-\infty}^{\infty} d\omega \hat{f}(\omega) \overline{\hat{\phi}(\omega)} e^{2\pi i \omega k} \quad (2.66)$$

$$= \int_0^1 d\omega \sum_{n \in \mathbb{Z}} \left(\hat{f}(\omega + n) \overline{\hat{\phi}(\omega + n)} e^{2\pi i (\omega + n)k} \right) \quad (2.67)$$

$$= \int_0^1 d\omega e^{2\pi i \omega k} \sum_{n \in \mathbb{Z}} \hat{f}(\omega + n) \overline{\hat{\phi}(\omega + n)}. \quad (2.68)$$

In the two last equalities, we first divided the integration interval in intervals of unit length, and then we used the periodicity of the complex exponential. The last equality is the inverse Fourier transform of the function

$$\sum_{n \in \mathbb{Z}} \hat{f}(\omega + n) \overline{\hat{\phi}(\omega + n)}, \quad (2.69)$$

which is equal to 0. Taking the Fourier transform, we conclude that

$$\sum_{n \in \mathbb{Z}} \hat{f}(\omega + n) \overline{\hat{\phi}(\omega + n)} = 0. \quad (2.70)$$

We can use the Cauchy-Schwarz inequality to argue that the above equation converges absolutely almost everywhere, which means that the convergence is independent of the order of summation. We now substitute (2.46) and (2.49) and split the sum into parts corresponding to odd and even n , quite similarly as we did above. The result is

$$m_f(\omega)\overline{m_0(\omega)} + m_f(\omega + 1/2)\overline{m_0(\omega + 1/2)} = 0. \quad (2.71)$$

Now, both m_0 and m_f are periodic with period 1. Furthermore, as a consequence of (2.62), $m_0(\omega)$ and $m_0(\omega + 1/2)$ cannot both be zero for a given ω (except on sets of zero measure). Thus, $m_f(\omega)$ and $m_0(\omega + 1/2)$ must be linearly dependent; there must exist a function λ with period 1, satisfying

$$m_f(\omega) = \lambda(\omega)\overline{m_0(\omega + 1/2)}, \quad (2.72)$$

$$\lambda(\omega) + \lambda(\omega + 1/2) = 0. \quad (2.73)$$

It is convenient to introduce a function ν with period 1 and write

$$\lambda(\omega) = e^{2\pi i\omega}\nu(2\omega). \quad (2.74)$$

With this definition, λ automatically satisfies (2.73). We have found

$$m_f(\omega) = e^{2\pi i\omega}\nu(2\omega)\overline{m_0(\omega + 1/2)}, \quad (2.75)$$

and we can write (2.46)

$$\hat{f}(\omega) = \nu(\omega)\underbrace{e^{2\pi i\omega/2}\overline{m_0(\omega/2 + 1/2)}}_{\hat{\psi}(\omega)}\hat{\phi}(\omega) \quad (2.76)$$

$$\equiv \nu(\omega)\hat{\psi}(\omega), \quad (2.77)$$

where we have introduced ψ , a candidate for a basis on W_0 . Since ν has period one, we can expand it in a (formal) Fourier series;

$$\hat{f}(\omega) = \nu(\omega)\hat{\psi}(\omega) = \sum_{k \in \mathbb{Z}} \nu_k e^{2\pi i\omega k} \hat{\psi}(\omega). \quad (2.78)$$

Now, since

$$\mathcal{F}^{-1}(\hat{\psi} \exp(-2\pi i\omega k)) = \psi(t - k), \quad (2.79)$$

we get upon taking the inverse Fourier transform of (2.78)

$$f(t) = \sum_{k \in \mathbb{Z}} \nu_k \psi(t - k). \quad (2.80)$$

This is exactly what we want; we have shown that we can represent f in W_0 by means of a linear combination of translates of ψ . The choice for ψ is not unique; if $\rho(\omega)$ is any function with period 1 satisfying $|\rho| \equiv 1$ almost everywhere, we can chose

$$\hat{\psi} = e^{2\pi i \omega/2} \overline{m_0(\omega + 1/2)} \hat{\phi}(\omega/2) \rho(\omega). \quad (2.81)$$

Now we need to show that $\psi(t - k)$ is indeed an orthonormal basis for W_0 . Without loss of generality, we assume $\rho \equiv 1$. This analysis is similar to what we have already done; we will use the definition of ψ , the periodicity of m_0 and a property of $\hat{\phi}$. We calculate the sum

$$\sum_{n \in \mathbb{Z}} |\hat{\psi}(\omega + n)|^2 = \sum_{n \in \mathbb{Z}} \left| m_0 \left(\frac{\omega + n + 1}{2} \right) \hat{\phi} \left(\frac{\omega + n}{2} \right) \right|^2 \quad (2.82)$$

$$\begin{aligned} &= |m_0(\omega/2 + n + 1/2)|^2 \sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega/2 + n)|^2 \\ &\quad + |m_0(\omega/2 + n + 1)|^2 \sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega/2 + n + 1/2)|^2 \end{aligned} \quad (2.83)$$

$$= |m_0(\omega/2)|^2 + |m_0(\omega/2 + 1/2)|^2 \quad \text{by (2.57)} \quad (2.84)$$

$$= 1 \quad \text{by (2.58),} \quad (2.85)$$

which is valid almost everywhere. Now we prove that $\psi_{0,k}$ are orthonormal.

$$\langle \psi_{0,0}, \psi_{0,k} \rangle_{L^2(\mathbb{R})} = \langle \mathcal{F}\psi_{0,0}, \mathcal{F}\psi_{0,k} \rangle_{L^2(\mathbb{R})} \quad \text{Parseval's Theorem} \quad (2.86)$$

$$= \langle \hat{\phi}, \hat{\phi} e^{-2\pi i \omega k} \rangle_{L^2(\mathbb{R})} \quad (2.87)$$

$$= \int_{-\infty}^{\infty} d\omega |\hat{\psi}|^2 e^{2\pi i \omega k} \quad (2.88)$$

$$= \int_0^1 d\omega e^{2\pi i k \omega} \sum_{n \in \mathbb{Z}} |\hat{\psi}(\omega + n)|^2 \quad (2.89)$$

$$= \int_0^1 d\omega e^{2\pi i \omega k} \quad \text{by (2.85)} \quad (2.90)$$

$$= \delta_{0,k}. \quad (2.91)$$

The last thing we need to show is that the formal series (2.80) is valid, i.e. we show that the Fourier coefficients ν_k in (2.80) satisfy

$$\sum_{k \in \mathbb{Z}} |\nu_k|^2 < \infty, \quad (2.92)$$

which is equivalent with the condition

$$\int_0^1 d\omega |\nu(\omega)|^2 \leq \infty. \quad (2.93)$$

We show this in three steps. First, we calculate

$$\int_0^1 d\omega |\nu(\omega)|^2 = \int_0^1 d\omega |\lambda(\omega/2)|^2 \quad (\text{by (2.74)}) \quad (2.94)$$

$$= 2 \int_0^{\frac{1}{2}} d\omega |\lambda(\omega)|^2. \quad (2.95)$$

We also have

$$\int_0^1 d\omega |m_f(\omega)|^2 = \int_0^1 d\omega |\lambda(\omega)m_0(\omega + 1/2)|^2 \quad (\text{by (2.72)}) \quad (2.96)$$

$$= \int_0^{\frac{1}{2}} d\omega |\lambda(\omega)m_0(\omega + 1/2)|^2 + \int_{\frac{1}{2}}^1 d\omega |\lambda(\omega)m_0(\omega + 1/2)|^2 \quad (2.97)$$

$$= \int_0^{\frac{1}{2}} d\omega |\lambda(\omega)m_0(\omega + 1/2)|^2 + \int_0^{\frac{1}{2}} d\omega |\lambda(\omega + 1/2)m_0(\omega + 1)|^2 \quad (2.98)$$

$$= \int_0^{\frac{1}{2}} d\omega |\lambda(\omega)|^2 (|m_0(\omega + 1/2)|^2 + |m_0(\omega)|^2) \quad (2.99)$$

$$= \int_0^{\frac{1}{2}} d\omega |\lambda(\omega)|^2. \quad (2.100)$$

where we have used (2.73) to get (2.99), and (2.62) to obtain (2.100). Therefore, we can conclude

$$\int_0^1 d\omega |\nu(\omega)|^2 = 2 \int_0^{\frac{1}{2}} d\omega |\lambda(\omega)|^2 \quad (2.101)$$

$$= 2 \int_0^1 d\omega |m_f(\omega)|^2 \quad (2.102)$$

$$= 2 \cdot \frac{1}{2} \sum_{k \in \mathbb{Z}} |f_k|^2 \leq \infty \quad (2.103)$$

where the inequality follows since $f \in W_0 \subset L^2(\mathbb{R})$. We conclude that the expansion (2.80) is valid, and the Theorem is proved.

3 SPECIAL TOPICS AND FURTHER GENERALISATIONS

Below, we first briefly discuss some practical aspects of the wavelet transform which may confuse the newcomer to wavelets, especially since they sometimes are touched lightly upon in the some of the more popular introductory texts. Then we point out an important generalisation of the basic multiresolution scheme we presented in the previous Section.

3.1 The Fast Wavelet Transform

In order to be able to reconstruct a function f from its wavelet transform, we assumed that it was an element in the scaling space V_0 , see (2.35). In many cases, f will not be in V_0 , but some larger space. The input to the FWT algorithm as defined in Theorem 1 is $c_0[k] = \langle f, \phi_{0,k} \rangle$, which means that we are actually calculating the wavelet transform of $P_{V_0}f$, the orthogonal projection of f onto V_0 given by

$$P_{V_0}f = \sum_k \langle f, \phi_{0,k} \rangle \phi_{0,k}. \quad (3.1)$$

If we are to use the wavelet transform to extract information from a signal $f \notin V_0$, we must make sure that V_0 is capable of describing all the details we are interested in, so that we do not lose essential information by taking the orthogonal projection of f on V_0 . The choice of V_0 could be done out of knowledge of the Nyquist frequency for our signal and physical considerations.

The input to the FWT algorithm is another cause of concern; when one is computing the expansion coefficients

$$c_J[k] = \langle f, \phi_{J,k} \rangle \quad (3.2)$$

$$d_j[k] = \langle f, \psi_{j,k} \rangle \quad j = 1, \dots, J \quad (3.3)$$

by means of the FWT as stated in (2.40) and (2.41), one needs to provide the initial coefficients $c_0[k]$ in order to start the algorithm off. Often, we cannot calculate them directly, either because it would be impractical, or more often because the function whose wavelet transform we seek is only known at some *sampled* values. Here, and in the following, we assume without loss of generality that the sampling period is unity. The usual way of dealing with this problem is to state that at a high resolution 0, the scaling functions are quite narrow, so $\phi_{0,k}(t) \approx \delta(t-k)$. Because of this approximation, we are somewhat justified in letting $c_0[k] \equiv f(k)$. However, this means that we are not calculating the wavelet transform of $f(t)$. Instead, we are calculating the transform of the function

$$f'(t) = \sum_l f(l)\phi(t-l) \quad (3.4)$$

which may or may not be a good approximation to the real $f(t)$. In particular, if $\phi(k) \neq \delta_{0,k}$, we have in general $f'(k) \neq f(k)$ for $k \in \mathbb{Z}$, which arguably seems unreasonable.

In [12] the authors make the case for *prefiltering* the samples before inputting them to the filter bank which implements the FWT. By a suitable choice of filter L , we can force the following to hold

$$f(k) = \sum_k (Lf)(l)\phi(k-l). \quad (3.5)$$

At present, it does not seem to be any consensus among researchers in the on how to handle this problem. The Wavelet Toolbox for MATLAB do not show any concern about the fact that they use samples as approximations for the initial inner products, and neither will we in this preliminary report. However, the problem should be addressed at a later stage, possibly in conjunction with any other preprocessing of the data such as noise removal and bandpass filtering to avoid aliasing.

3.2 Time invariance

For all the nice properties of the wavelet transform, it is an unfortunate fact of life that it is not time invariant. With this we mean that there is in general no simple relationship between the wavelet transform of a function and the transform of a shifted version of the same function. On the other hand, the Fourier transforms of two such functions differ only in that the latter would be modulated by a sinusoidal component. It is clear that time invariance is something which is desirable if we want to use wavelets for detecting signals occurring at unknown times; if we have a detector capable of detecting a signal occurring at time $t = 12s$, but which is not capable of detecting the same signal occurring 3s later, then it is tempting to say that we have a badly designed detector. To substantiate the claim that translation invariant detectors perform better, see e.g. [14],[15]. Several methods have been proposed to deal with this problem. Most utilise the following property in some form.

Theorem 3 *Let S be the right shift operator such that $Sf(t) = f(t+1)$, and let T_e be a transform as in (2.36). Then, for $e = \phi$ or $e = \psi$ and $q = q_0 + 2^j q_1$, $q_0 = 0, 1, \dots, 2^{j-1}$ we have*

$$\left(T_e S^{q_0 + q_1 2^j} f \right) (j, k) = S^{q_1} \left(T_e S^{q_0} f \right) (j, k). \quad (3.6)$$

This property follows by simple changes of integration variables.

Theorem 4 *To compute the J -wavelet transforms of all N shifts of a sequence of length $N = 2^n$ takes $O((J+1)N)$ storage space. This can be performed in $O(N \log_2 N)$ time.*

The reasoning behind proof is as follows. Since q_0 can take 2^j values at level j , Theorem 3 states that we need to calculate 2^j wavelet transforms. Now, the the number of coefficients in the DWT of a sequence of length N is halved at each level. Then, at level j the total number of coefficients we need to store equals the number of coefficients

in the DWT at that level multiplied with the number of DWTs we must calculate; $N2^{-j} \cdot 2^j = N$. We have J detail levels, so to store all detail coefficients, we need JN storage space. At the last level J we also must have the approximation structures of 2^J shifts, each having $N2^{-J}$ approximation coefficients. In order to store the entire wavelet decomposition structure of all possible shifts, we thus need $N(J + 1)$ coefficients. As for the time estimate, we use the fact that the wavelet transform is an $O(N)$ algorithm, and that it is applied to 2^j sequences of length $N2^{-j}$ at each level. Since $J \leq \log_2 N$, the theorem follows.

There exists other translation invariant schemes, for example as developed in [15].

3.3 Wavelet packets

This section deals briefly about an important generalisation of wavelets. The best way to understand wavelet packets is to look at Figure 2.2, and reason as follows; the coefficients $d_1[k]$ represent the highest details in our signal. If the parts of the signal we are interested in belongs to the approximation space $V_1[k]$, it makes perfectly sense to further decompose these coefficients, as in Figure 2.2. However, if the first wavelet space W_1 contains important parts of the signal, possibly different high frequency components, then we should probably decompose it in order to separate the different components. In short, it may be valuable to consider the *full* binary tree decomposition structure as in Figure 3.1. This gives us more freedom in choosing the decomposition structure of a signal. The extra freedom can be used to good effect; in the literature,

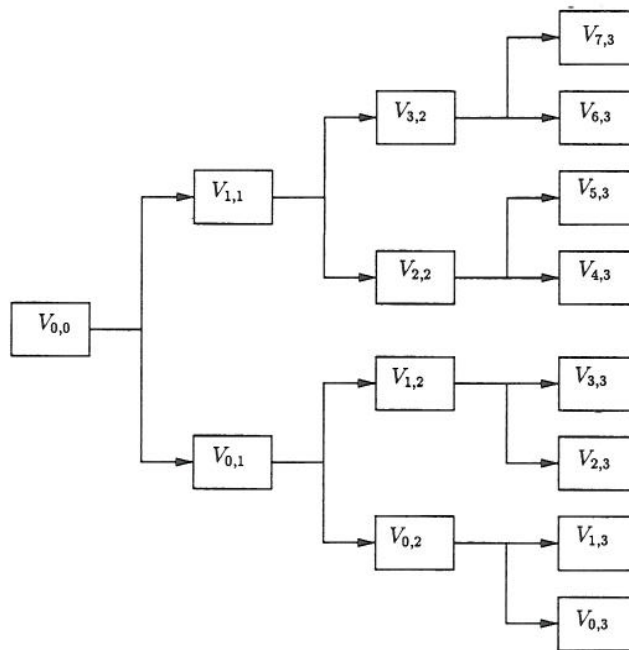


Figure 3.1 Schematical representation of how the initial space V_0 is decomposed using a three-stage wave packet decomposition.

wavelet packet detectors generally outperform traditional wavelet detectors [14]. The current trend in the signal processing literature also seems to be in favour of wavelet packet-based detectors.

In the wavelet packet decomposition structure, we decompose at each level j our initial space V_0 into 2^j spaces. We identify the decomposition space not only by a scale index j , but also by a *frequency index*. These new spaces satisfy

$$V_{j,n} = V_{j+1,2n} \oplus V_{j+1,2n+1}, \quad n = 0, 1, \dots, 2^j - 1. \quad (3.7)$$

It is clear from Figure 3.1 that wavelets are a special case of the more general wavepacket formulation; $V_{0,j} = V_j$ and $V_{1,j} = W_j$.

There are several strategies for choosing the decomposition structure. One method which has received much attention, especially in the field of data and image processing is Coifman's Best Basis Algorithm [17]. The idea behind the algorithm is to minimise a *cost function* which is small when most of the coefficients in the decomposition are small. One performs a full wavepacket decomposition and out of all possible decompositions, one finds the one which defines a global minimum in the cost function.

In our application, detection of signals whose waveform is approximately known, it may be most efficient to choose a structure which concentrates the energy of the transients in as few coefficients as possible. This means that we can choose the decomposition structure *a priori*. A related technique is used to with good results in [16].

4 TRANSIENT DETECTION AND CHARACTERISATION

As stated in the introduction, one of the goals of the CHESSE-programme is to develop a sensor system to monitor the strain on the hull of a ship. During operation, data will be continuously fed into a data processing system. On the basis of these data, the system decides in real time whether a transient has occurred. If a transient is detected, the system may characterise the transient by estimating time of arrival, duration, amplitude, frequency, and possibly other parameters as well. The results should be presented to the user, that is to the Captain of the ship. On the basis of this output, the captain should be able to operate the ship in such a manner that there is minimal risk of damage to the structure.

Detecting whether a signal is present in a noisy background is a classical problem with a wide area of applications. Consequently, there exists an extensive literature on the area. A good survey article is [21], upon which much of this chapter is based.

4.1 Preliminary considerations

We designate the transient we wish to detect by $s(t)$, and the measured signal from the sensors by $x(t)$. Assume that $s(t)$ has support $[0, T]$, and that we observe x for $t \in M$, where $[t_i, t_i + T] \subseteq M$. Then we can formulate the following hypothesis test.

$$\begin{aligned} H_0: x(t) &= n(t) \\ H_1: x(t) &= s(t - t_i; \theta) + n(t), \end{aligned} \tag{4.1}$$

where $t \in M$ and $n(t)$ is noise, and we assume that we know the time of arrival t_i and the characteristic duration T of the transient. The vector θ represent unknown parameters, e.g. frequency, damping, and amplitude. To simply assume that the noise is zero mean white noise is convenient, but not reasonable. The measured strain on the hull is among other things dependent on sensor location, on the direction and speed of the ship relative to the oncoming waves. In spite of these considerations, we will assume as a first approximation that the noise is indeed zero mean white noise. As we shall see, we will get reasonable results with this crude approximation. However, further studies on noise estimation, whitening and removal is recommended.

For a given observation of x with $t \in M$, we need a formal way of choosing between H_0 and H_1 . This is done by comparing a statistic with some threshold level. We call this statistic a *detector*. In its simplest form it is a 'black box', where data $x(t)$ is input, and the single output $y(t)$ is used to distinguish between the hypotheses. To make a good statistic, it should be chosen in such a way that the output under the two competing hypotheses are, on average, very different. This requirement will be made precise in Section 4.2. We also state the obvious; the detector cannot detect a transient which has not yet arrived. In filter jargon, the detector has to be *causal*, i.e. it must not be dependent on future data.

4.2 Matched filter

We would like to make our detector optimal, in some sense. This is commonly done by running the input signal x through a linear filter, which is designed to maximise the output (power) signal-to-noise ratio (SNR_o) *when a signal is present*, that is, under H_1 . Such a filter is called a *matched filter*. We designate the impulse transfer function of this filter by h_{MF} . If we input $x(t) = s(t) + n(t)$, the output is given by

$$x * h_{MF} = s * h_{MF} + n * h_{MF}. \quad (4.2)$$

The output signal to noise ratio SNR_o is defined as

$$\text{SNR}_o = \frac{|s * h_{MF}(T)|^2}{\text{E}[|n * h_{MF}|^2]}. \quad (4.3)$$

When the noise is white, it follows from Schwartz's inequality applied to the above equation that the optimum impulse response h_{MF} is proportional to a delayed time-reversed version of s . This is a well-known result in signal processing theory, and we write

$$h_{MF}(t) = \begin{cases} \text{const} \cdot s(T - t) & t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \quad (4.4)$$

The disadvantage with this matched filter technique is that we need to know the waveform of the transient we wish to detect a priori. In our case, the transients must be assumed to vary in frequency and damping, and possibly in other respects as well.

A standard method for estimating such unknown parameter is to run our signal through a *bank* of parallel filters, each filter being matched to a transient with slightly different parameters. The estimate of the parameter at a given time would then be the parameter corresponding to the filter with largest output.

4.2.1 Time domain

In the time domain, the filter output is given by convolution with the transfer function (4.4);

$$y(t) = \int dt x(\tau) h_{MF}(t - \tau) \quad (4.5)$$

$$= \int dt x(\tau) s(T - t + \tau) \quad (4.6)$$

$$= \int_0^T dt x(\tau + t - T) s(\tau). \quad (4.7)$$

The last equality follows by making a change of variables, and noting that s has compact support. In discrete time, we have similarly

$$y(t) = \sum_{\tau=0}^T x(\tau + t - T)s(\tau). \quad (4.8)$$

We will now describe how the filtering can be done in the transform domain.

4.2.2 Transform domain

As proposed by Eriksen in [1] and further developed by Urnes in [2], the matched filtering can be done in the transform domain when we have a decomposition by means of a tight frame. In that case, the frame condition (2.12) states that for some constants A, B ,

$$A\langle f, f \rangle_H = \langle Gf, f \rangle_H = B\langle f, f \rangle_H. \quad (4.9)$$

This implies that the metric operator is given by $G = A\text{Id}$. But we also have $\langle Gf, f \rangle_H = \langle Tf, Tf \rangle_{L^2(\mu)}$. In terms of the norms,

$$\|Tf\|_{L^2(\mu)}^2 = A\|f\|_H^2. \quad (4.10)$$

Since we are working in Hilbert spaces, we can use the polarisation identity to construct a valid inner product on $L^2(\mu)$ from its norm, which is proportional to the norm on H , which in turn is expressible in terms of the inner product on H . In that way, the fact that the norms on the two spaces are equivalent enables us to get an equivalence of the *inner products* on the two spaces. This is precisely a Parseval's Theorem for tight frames.

Theorem 5 (Parseval) *Let H be a Hilbert space with measure μ , and let $\{e_m\}_{m \in M}$ be a tight frame in H indexed by M . The transform $T : H \rightarrow L^2(\mu)$ defined by $T = \langle f, e_m \rangle_H$ satisfies, for any $f, g \in H$*

$$\langle Tf, Tg \rangle_{L^2(\mu)} = A\langle f, g \rangle_H. \quad (4.11)$$

This Theorem can be used to prove the following result;

Theorem 6 (Wavelet domain matched filtering) *Let the matched filtering of a signal x with a reference signal $s(t)$ be defined as in (4.5), so that the output is given by*

$$y(t) = \int_0^T dt x(\tau + t - T)s(\tau). \quad (4.12)$$

This output $y(t)$ can equivalently be computed in the wavelet transform domain when we are using a orthonormal wavelet J -level wavelet decomposition. It is then given by the equation

$$y(t) = \sum_{k=-\infty}^{\infty} (\mathbb{T}_{\phi}x)(J, k) \overline{(\mathbb{T}_{\phi}S^{q_0}s)(J, k + q_1)} + \sum_{k=-\infty}^{\infty} \sum_{j \in D} (\mathbb{T}_{\psi}x)(j, k) \overline{(\mathbb{T}_{\psi}S^{q_0}s)(j, k + q_1)} \quad (4.13)$$

where $D = \{1, 2, \dots, J\}$ and $T - t = q_0 + q_1 2^j$ as defined in Theorem 3.

The proof goes as follows. If we have an orthonormal basis, $A = 1$, and we can then write the filter output (4.6) as

$$y(t) = \int_{-\infty}^{\infty} d\tau x(\tau) s(T - t + \tau) \quad (4.14)$$

$$= \langle x, S^{T-t}s \rangle_{L^2(\mathbb{R})} \quad (4.15)$$

$$= \langle \mathbb{T}x, \mathbb{T}S^{T-t}s \rangle_{l^2(\mathbb{R}^2)}, \quad (4.16)$$

where the last equality follows from Theorem 5 and S is the shift operator. The Theorem then follows by writing out the inner product (4.16) explicitly.

Because of Parseval's Theorem, the output of the wavelet based matched filtering is identical with that of the time-domain matched filter. However, if we restrict the set D to contain only the scales where transient has significant amounts of energy, we have a potentially better detector; this approach separates different components of the signal into different scales, thereby effectively increasing the signal-to-noise ratio. By combining this detector with noise reduction and wavelet packet best basis algorithms, one should be able to enhance its performance even more.

4.3 Non-parametric detectors

In the case where we do not know the waveform of the transient, we must retort to non-parametric methods. In [18] it is claimed that in this case, there does not exist a "uniformly most powerful approach". If we merely are interested in detecting whether there is a transient present in an interval, an intuitive statistic would be the energy in that interval. This detector is rather crude; it does not indicate if one or many transient arrived, nor does it provide much information about their characteristics. Another promising class of detectors are based on the magnitude or energy in the wavelet coefficients.

4.3.1 Energy detector

If the signal we are trying to detect has a duration T , we can define the detector

$$y(t) = \|x\|_{L^2([t-T, t])}^2. \quad (4.17)$$

Since we are working with samples, the detector

$$y(t) = \|x\|_{l^2([t-T, t])}^2 \quad (4.18)$$

is more appropriate. In both cases we use only data up to time t .

For tight frames, there is a simple relation between the energy in the time and transform domains, $\|x\|_{L^2(\mathcal{R})}^2 = A^{-1} \|x\|_{l^2(\mathcal{Z}^2)}^2$. This observation could be used as a starting point for defining an energy detector in the wavelet domain. However, want an energy detector on a finite time interval, and it is not clear what the corresponding interval in the transform-domain would be. We leave this detector for future work,

4.3.2 Maximum transform coefficient detector

The wavelet coefficients can be interpreted as a measure on the correlation, or similarity, between the analysing wavelet and the signal itself. A large detail (approximation) coefficient at a given time k and scale j indicates a relative large degree of similarity between the wavelet $\psi_{j,k}$ (scaling function $\phi_{j,k}$) and the signal x . By this argument, we can base a detector on

$$(T_e S^\tau x)(j, k) = \int_{-\infty}^{\infty} d\tau x(\tau + q) \overline{e_{j,k}(\tau)}, \quad (4.19)$$

where $e = \phi$ or $e = \psi$. Using Theorem 3 and letting $q = q_0 + q_1 2^j$, we define the maximum wavelet coefficient detector as

$$y_j(q) = (T_\psi S^{q_0} x)(j, k + q_1). \quad (4.20)$$

By restricting the j to only take the values which corresponds to the scales where most of the energy or our transients are located, it is possible to 'tune' this detector so as to only give high values for signals which are similar to the ones we wish to detect.

5 PRACTICAL EXAMPLES

The process of designing the signal processing system has just about started up. Therefore, results presented in this section are very much preliminary, and serve mainly as demonstrations of the methods we are developing.

5.1 Matched filter

We present examples of matched filtering in both domains. At the top of Figure 5.1 we show an excerpt of a long signal, and identify one very large transient at time $t = 339s$. Below the signal are the magnitude of the output of two matched filter detectors. Both are matched to the same transient, but one of them is performing the filtering in the wavelet domain, using only the the scale where most of the transient's energy is located.

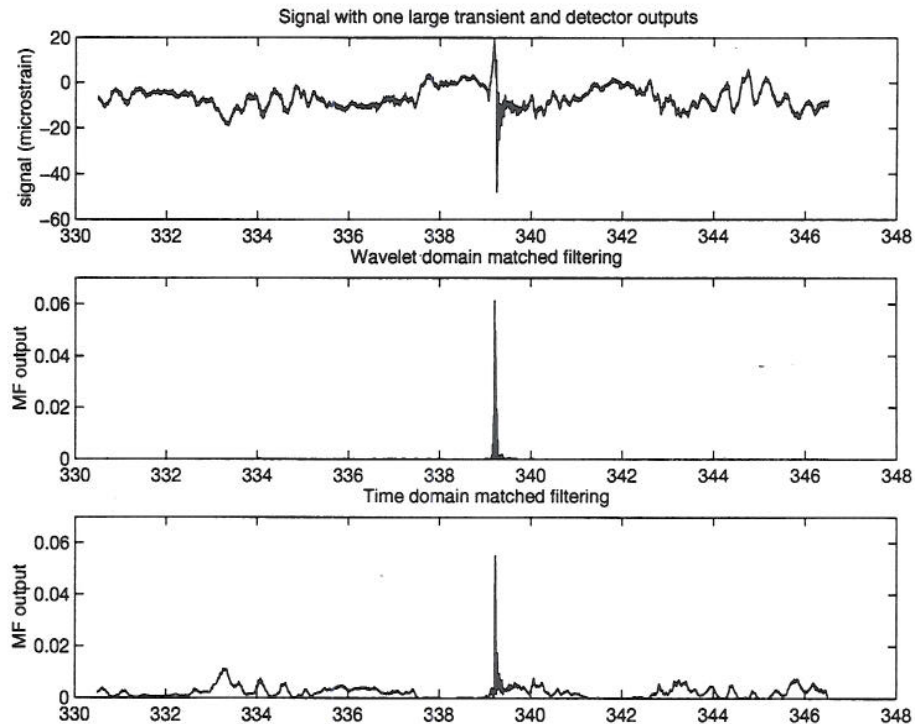


Figure 5.1 A signal with a large transient and detector output based on matched filtering in the time domain wavelet domain using two scales only.

We see that both detectors perform well, in the sense that they give a sharp spike at the instant the transient occurs. As we discussed in Chapter 4, our detectors will be causal, and their output at time $t+T$ will give us an indication as to whether a transient arrived at time t or not. In Figure 5.1 we have shifted the matched filter output T seconds to the left, to better compare the time of arrival to the maximum matched filter output. most of the energy in the transient is located. We have also scaled the plots, so that the spike has the same height. Comparing the signal-to-noise-ratio in the to MF plots, we observe that it is several orders of magnitude less for the wavelet-based matched filter. This is because a wavelet decomposition separates different phenomena into separate scales.

If we performed the matched filtering in the wavelet domain, using *all* scales, our

detector output would be equivalent to the time domain output. In Figure 5.2 we illustrate this by plotting the magnitude of difference of the detector outputs from

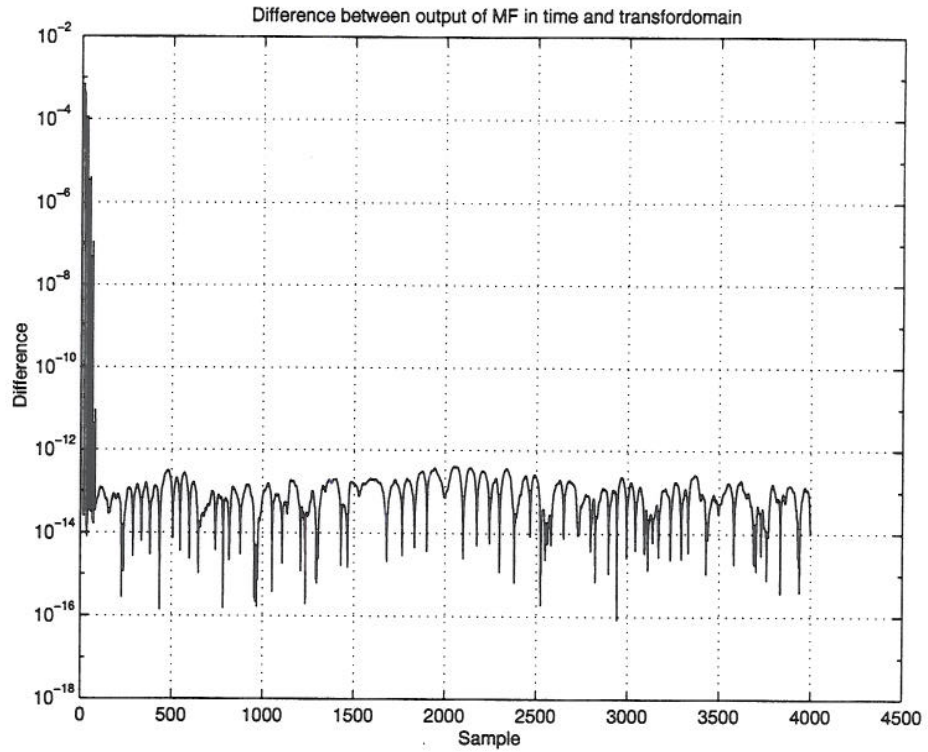


Figure 5.2 The absolute value of the difference of matched filter detector outputs performed in time and wavelet domain.

timed domain and wavelet domain matched filtering. For the most part it is a few orders of magnitude above the machine accuracy $\epsilon = 2 \cdot 10^{-16}$. The relatively large discrepancy in the beginning is probably due to how the MATLAB Wavelet Toolbox handles sequences with lengths that are not a power of 2.

6 SUMMARY AND FURTHER WORK

6.1 Summary

In this report we have reviewed the basic theory of dyadic orthonormal wavelet expansions which are computable by a fast algorithm, and discussed several generalisations of the basic wavelet transform. Then we looked at how we could use these techniques for detecting imperfectly known transient signals occurring at unknown times.

In the Section 2 we showed how a signal could be transformed by means of inner product with a possibly linearly dependent set of functions. This approach was shown to provide a common framework for both the short time Fourier transform and the discrete and continuous wavelet transform. We defined the concept of multiresolution and using this we showed how the wavelet transform could be performed efficiently by a filter bank.

Section 3 discussed selected topics in some more detail, and pointed out how the wavelet transform could be generalised with bi-orthogonality, multiwavelets, M -band or wavepacket decompositions.

Methods for detecting and characterising transients were addressed in Section 4, and we constructed detectors based upon matched filtering in both the time and time-scale domain. We saw how we could estimate unknown parameters in the transient by passing the signal through a filter bank matched to transients with different parameters. The matched filters were based upon knowing or having a good model of the transients we were seeking. Detectors which were not based upon any assumptions regarding the waveform of the transients were presented; one was based on the energy of the signal in the time domain, another was based on the energy in the transform domain. The latter was the more promising, because it allowed us to separate the energy of phenomena which occurred at different scales, or frequencies. It is also possible to use base a detector directly on the wavelet transform coefficients.

6.2 Further work

Here we point out some of the areas which should be investigated further.

A wavelet packet based maximum wavelet detector shows good promise for our application. By computing a good basis, either as we go along, or using a fixed basis based on an analysis of the transients we wish to detect, we are able to concentrate the energy of the signal in a few coefficients.

Another way to think of this is to regard the ensemble of all possible transients as basis functions spanning a subspace of the space spanned by all possible signals. A good maximum wavelet coefficient detector should use wavelets that 'mostly' are orthogonal to this subspace. That makes the inner product between elements of this subspace and most of the wavelets small - except for a characteristic few wavelets. This point-of-view is related to the sub-space analysis concept, which has received a certain degree of attention in the system identification community the last few years. Most of the literature used so far in the CHESS project has its origin in the signal

processing community. One should investigate how problems related to transient detection and characterisation are handled in other engineering fields. In particular, the system identification community and their journals could prove to be valuable sources of information.

As for practical implementation work, it is clear that some of the algorithms has to be hard coded in Fortran, C++ or some other low level language. The main bottle neck in our Matlab programmes is Matlab's infamously slow for-loops. We can reduce the number of these loops by better algorithms, but where we have to do non sequential memory accessing, they are hardly to avoid.

A preliminary study indicates that gains in execution time when the for-loops are relegated to a machine code programme are about two to three orders of magnitude.

List of symbols and acronyms

Symbol	Meaning
$1_T(x)$	Characteristic function on T . Equals unity if $x \in T$, zero otherwise.
*	Convolution; Adjoint.
$\downarrow 2$	Downsampling by two, removes every second component of a sequence.
\hat{f}	Fourier transform of f , $\hat{f}(\omega) = \int dt f \exp(2\pi i\omega t)$.
$\langle \cdot, \cdot \rangle$	Inner product.
\tilde{f}	STFT of f ; general transform of f .
\oplus	Orthogonal direct sum.
$\psi(t)$	Mother wavelet.
$\psi_{j,k}(t)$	Scaled and translated wavelet, $\psi_{i,j}(t) = 2^{-j/2}\psi(2^{-j}t - k)$.
$\psi_k(t)$	$\psi_{0,k}(t)$.
$\phi(t)$	Scaling function.
$\phi_{j,k}(t)$	Scaled and translated scaling function, $\phi_{j,k}(t) = \phi(2^{-j}t - k)$.
$\phi_k(t)$	$\phi_{J,k}(t)$.
\bar{z}	Complex conjugate of z .
a	Continuous scaling parameter.
b	Continuous shift parameter.
$c_{J,k}, c_J[k]$	Level J approximation coefficients.
$d_{j,k}, d_j[k]$	Level j detail coefficients.
\mathcal{F}	Fourier transform operator.
H	Hilbert space.
h_{MF}	Impulse transfer function of a matched filter.
Id	Identity operator.
$L^2(\mu)$	Space of functions which are quadratically integrable with measure μ .
$n(t)$	Noise.
S	Left shift operator, $Sf(n) = f(n+1)$; synthesising (inverse) operator.
$s(t)$	Signal, transient we are looking for.
T, T_ϕ	Transform, transform with respect to the function ϕ .
V_j	Scaling space j .
$V_{j,n}$	Decomposition space at scale j , frequency n .
W_j	Wavelet space j .
$w(t)$	Window function.
$x(t)$	Input to detector or filter.
Acronym	Meaning
DFT	Discrete Fourier transform.
DWT	Discrete wavelet transform.
FWT	Fast wavelet transform.
MF	Matched filter.
SNR _o	Output signal-to-noise ratio.
STFT	Short-time Fourier transform.
QMF	Quadratic mirror filters.

References

- [1] R. Eriksen, "Realtime system for structure monitoring of ships using wavelets for transient characterization and detection", FFI/RAPPORT-96/03149, 1996.
- [2] E. Urnes, "Wavelets and their use in the detection and characterization of transient signals", FFI/RAPPORT-97/03066, 1997.
- [3] F. Hlawatsch and G. F. Bodreaux-Bartels, "Linear and Quadratic Time - Frequency Signal Representations," *IEEE Signal Processing Magazine*, 1992.
- [4] Cohen, L. *Time Frequency Analysis*, Prentice Hall, 1995.
- [5] B. Picinbono, "On Instantaneous Amplitude and Phase of Signals", *IEEE, Transactions on Signal Processing*, **45** (1997), pp. 552-560.
- [6] G.Kaiser, *A Friendly Guide To Wavelets* Birkhäuser, Boston, 1994.
- [7] N.Young, *An introduction to Hilbert space*, Cambridge University Press, Cambridge, 1988.
- [8] I. Daubechies, "The Wavelet Transform, Time-Frequency Localization and Signal Analysis", *IEEE, Transactions on Signal Processing*, **36** (1990), pp. 961-968.
- [9] I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelphia, 1995.
- [10] A. Cohen and R.D. Ryan, *Wavelets and Multiscale Signal Processing*, Chapman & Hall, London, 1995.
- [11] C. S. Burrus, R.A. Gopinath, H. Guo, *Introduction to Wavelets and Wavelet Transforms*, Prentice Hall, to be published 1997.
- [12] G. Strang, T.Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Wellesley 1996.
- [13] G. Beylkin, "On the Representation Of Operators in Bases of Compactly Supported Wavelets", *Siam J. Numer. Anal.*, **6** (1992), pp. 1716-1740.
- [14] S. Del Marco, J .Weiss, "M-band wavepacket-based transient signal detector using a translation invariant wavelet transform", *Optical Engineering*, **33** (1994), pp. 2175-2182.
- [15] S.Del Marco, J. Weiss, "Improved Transient Signal Detection Using a wavepacket-based detector with an Extended Translation-Invariant Wavelet Transform", *IEEE, Transactions on Signal Processing*, **45** (1997), pp. 841-849.
- [16] R. E. Learned, "A Wavelet Packet Approach to Transient Signal Classification", Department of Electrical Engineering and Computer Science, M.I.T., *Preprint*.
- [17] R. Coifman, M.V. Wickerhauser, "Entropy-based algorithms for best basis selection", *IEEE Transactions on Information Theory*, **38** (1992), pp. 713-718.
- [18] M.Frisch, H.Messer, "The Use of the Wavelet Transform in the Detection of an Unknown Transient Signal", *IEEE Transactions on Information Theory*, **38** (1992), pp. 892-897.
- [19] M.Frisch, H.Messer, "Transient Signal Detection Using Prior Information in the Likelihood Ratio Test," *IEEE Transactions on Signal Processing*, **41** (1993), pp. 2177-2192.
- [20] M.Misiti, Y.Misiti, G.Oppenheim,J-M.Poggi, *Wavelet Toolbox User's Guide*, The MathWorks Inc., Natick, Mass., 1996.
- [21] G.Turin, "Introduction to Matched Filters", *IEEE Transactions on Information Theory*, 1960, pp. 311-329.
- [22] R.G.Bartle, *The elements of Integration and Lebesgues Mesasure*, John Wiley & Sons, Inc., New York, 1995.

A Measure theory

Here we present some basic results from measure theory and the Lebesgue integral. Although these considerations are not truly essential to grasp the basics of wavelet analysis, they provide the formal justification of much of that which we have done. For example, one frequently needs to change the order of integration, which is allowed if the conditions of *Fubini's theorem* are satisfied. Another advantage is that we can use the Lebesgue framework to represent integrals and sums with the same symbol.

In order to state some key results, we need a few technical definitions. The *extended real numbers* are the set $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. With this we mean that $\bar{\mathbb{R}}$ consists of \mathbb{R} and the two *symbols* (not numbers) $-\infty, \infty$.

Definition 4 (Algebras and σ -algebras) *A family \mathbf{A} of subsets of a set X is called a algebra if the following hold:*

1. $\emptyset, X \in \mathbf{A}$.
2. If $E \in \mathbf{A}$ then the complement E^c satisfies $E^c \in \mathbf{A}$.
3. If $(E_n)_{n=1}^N$ is sequence of sets in \mathbf{A} , then $\cup_{n=1}^N E_n \in \mathbf{A}$.

If $(E_n)_{n=1}^\infty$ is sequence of sets in \mathbf{A} , and $\cup_{n=1}^\infty E_n \in \mathbf{A}$, we call \mathbf{A} a σ -algebra.

We call an ordered pair (X, \mathbf{A}) consisting of a set X and a σ -algebra \mathbf{A} of subsets of X a *measurable space*. An element in \mathbf{A} is called a \mathbf{A} -measurable set. When the σ -algebra is fixed we will simply say that it is measurable.

Definition 5 (Measure) *A measure μ is function from an algebra \mathbf{A} of subsets of a set X to the extended real numbers,*

$$\mu : \mathbf{A} \rightarrow \bar{\mathbb{R}} \tag{A.1}$$

such that

1. $\mu(\emptyset) = 0$
2. $\mu(E) \geq 0 \quad \forall E \in \mathbf{A}$
3. If (E_n) is any pairwise disjoint sequence so that $\cup_{n=1}^\infty E_n \in \mathbf{A}$, then

$$\mu(\cup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n). \tag{A.2}$$

We will now define the concepts needed to define the integral.

Definition 6 (Simple function) *A measurable real-valued simple function takes only a finite number of values. It can be represented as*

$$\phi(x) = \sum_{j=1}^n a_j \mathbf{1}_{E_j}(x) \quad (\text{A.3})$$

where E_j

Definition 7 (Integral of a simple non-negative function) *The integral of a simple non-negative function ϕ with respect to the measure μ is defined as the extended real number*

$$\int d\mu \phi = \sum_{j=1}^n a_j \mu(E_j). \quad (\text{A.4})$$

We note that the integral may be infinity. The integral of a function f is introduced as the supremum of integrals of piecewise constant functions ϕ satisfying $\phi \leq f$.

Definition 8 (Integral of a non-negative function) *Let f be a non-negative function f belonging to the measure space (X, \mathbf{A}) , and let $E \in \mathbf{A}$. The integral of f with respect to the measure μ is defined as*

$$\int d\mu(x) f(x) = \sup_{\phi} \int d\mu(x) \phi(x) \quad (\text{A.5})$$

where the sup is taken over all ϕ satisfying $0 \leq \phi(x) \leq f(x)$ for all $x \in X$.

Now we let $X = \mathbb{R}$ and define the algebra \mathbf{A} to be the family of finite unions of the form

$$(a, b), \quad (-\infty, b], \quad (a, \infty), \quad (-\infty, \infty). \quad (\text{A.6})$$

We can define a measure on this algebra by

$$\mu((a, b]) = g(b) - g(a), \quad (\text{A.7})$$

where g is a monotonous right continuous function. Unfortunately, \mathbf{A} thus defined is not a σ -algebra, and, consequently not measurable. We work our way around this problem by extending \mathbf{A} to an σ -algebra $\mathbf{A}^* \supset \mathbf{A}$, and extending the measure μ to an *outer measure* μ^* which satisfies $\mu(E) = \mu^*(E)$ for all $E \in \mathbf{A}$. According to the extension theorems of Carathéodory and Hahn, this can be usually be done in an unique way, see [22]. We call μ^* the *Lebesgues-Stieltjes measure* generated by g . The

σ -algebra \mathbf{A}^* is usually larger than necessary. Therefore, one often works with *smallest* σ -algebra that contains \mathbf{A} , but we need not consider this.

The definition of the integral was only valid for non-negative functions. It can be extended to all functions by splitting f into positive and negative parts so that $f = f^+ - f^-$, and then integrating.

The usual measure employed is obtained by picking $g(x) = x$ in (A.7). It is readily seen that this is the *length* of a interval. For brevity, we often write this as $d\mu = dx$. In higher dimensional spaces, we can define measures by taking a product of the measures on each of the constituting single dimensions; for example, the continuous wavelet transform is based on the two-dimensional measure $d\mu = da db/a^2$.

Measure theory also provides a seamless transition from continuous integrals to discrete sums, as we shall now see. For this, we need a measure on the natural numbers \mathbb{N} , which we will call *the counting measure*. The σ -algebra we need is the set of all subsets of \mathbb{N} , and let E be an element in this algebra. Define $\mu(E)$ be the number of elements in E when E is finite, and if E is infinite, we assign to $\mu(E)$ the symbol ∞ . With this measure,

$$\int d\mu(x) a(x) = \sum_{n=-\infty}^{\infty} a(n), \quad (\text{A.8})$$

as promised.

B Computer Programmes

Some of the computer programmes we have developed during the work are listed here. A short documentation is also included.

B.1 Short documentation

We represent the shift invariant wavelet transform by an array with several fields, as indicated in figure B.1. The approximation coefficients are stored in W.cA, and

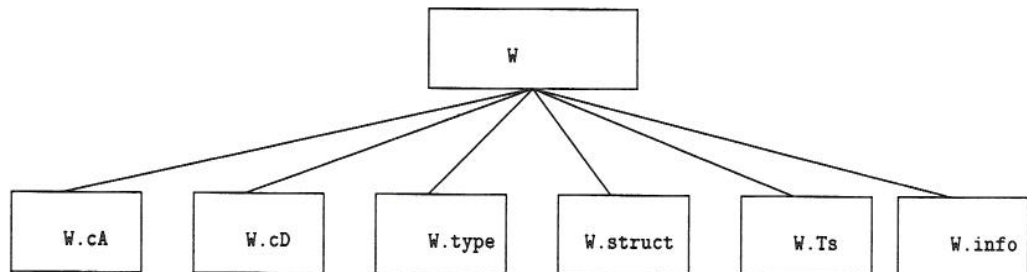


Figure B.1

Graphical representation of the data structure where the discrete wavelet transform is stored.

the detail coefficients are stored in W.cD. Using the notation of Theorem 3, we find

the detail coefficient $(TS^{q_0+q_1 2^j} f)(j, k)$ in $W.cD\{q_0, j\}(k+q_1)$ and the approximation coefficient $(TS^{q_0+q_1 2^j} f)(J, k)$ in $W.cD\{q_0\}(k+q_1)$. The other fields store the wavelet structure L as defined in the Wavelet Toolbox [20], the type of wavelet used, sampling interval and an arbitrary user supplied text string.

B.2 Programme listings

- Time invariant Wavelet Transform

```
function out=tiwt(x,J,type,Ts,info)
% Translation Invariant Wavelet Transform

if nargin <3
    error('too few input arguments')
elseif nargin > 5
    error('too many input arguments')
else
    W.cD=cell(2^J,J);
    W.cA=cell(2^J,1);

    for q0=0:2^J-1 % For all shift necessary
        qf= qfactor(q0+1); % See doc.
        [C,L]=wavedec(rshift(x,q0),J,type);
        if q0==0 Lout=L; end
        % q0=0:2^J-1, but Matlab's arrays starts with 1.
        % Therefore, the
        % 'mathematical correct' q0 is replaced with q0+1
        % in array indices.
        % Get cAJ first, then cDj (j=J,..1); Totally J+1 coefs.

        W.cA{q0+1}=C(1:L(1)); % get cAJ

        for j=1:J-qf
            % qf, because at we don't need all scales at all shifts
            k=J+1-j;
            % W.cD{q0+1}{j} shall contain cD(J+1-j),qv. Wavedec
            W.cD{q0+1,j}=detcoef(C,L,k);
        end
    end
    W.type={type};
    W.struct=Lout;

    if nargin==4
        W.Ts=Ts; % Sampling period
    elseif nargin==5
        W.info=info; % Comments etc.
    end
end
end
```

```
out=W;
```

- The ordinary wavelet transform

```
function out=wt(x,J,type,Ts,info)
% Wavelet Transform

if nargin <3
    error('too few input arguments')
elseif nargin > 5
    error('too many input arguments')
else
    W.cD=cell(J);
    W.cA=cell(1);

    [C,L]=wavedec(x,J,type);
    % Get cAJ first, then cDj (j=J,..1); Totally J+1 coefs.
    W.cA=C(1:L(1)); % get cAJ
    for j=1:J
        k=J+1-j;
        % W.cD{j} shall contain cD(J+1-j),qv. Wavedec
        W.cD{j}=detcoef(C,L,k);
    end
end
W.type={type};
W.struct=Lout;

if nargin==4
    W.Ts=Ts; % Sampling period
elseif nargin==5
    W.info=info; % Comments etc.
end
end

out=W;
```

- Detector

```
function y=detectb(x,bank,Wx,Wbank,which_d_scales)
% performs time and wavelet based matched filtering
% Amund Solvi Bremer
% 7.08.1997
[transN Nfilters]=size(bank)

%--time domain mathed filtering

y.tdmf=zeros( length(x), Nfilters); % preallocation
for i=1:Nfilters % mathed filter for each transient
    y.tdmf(:,i)=mfcorrelator(x,bank(:,i));
```

```
end
```

```
%--wavelet domain matched filtering
```

```
y.wdmf=zeros( length(x), Nfilters); % preallocation
for i=1:Nfilters
    y.wdmf(:,i) = wdmf(Wx,Wbank(i),which_d_scales);
end
```

- Wavelet based matched filtering

```
function y=wdmf(Wx,Ws,which_d_scales)
% Wavelet Domain Matchet Filtering. y=wdmf(Wx,Ws,which_d_scales)
% Wx- Wavelet transform of the signal x
% Ws- Time invariant WT of the signal s we are looking for
% whichd_scales = [0 1 0 0 1] (e.g.)
% -> filter at detail levels two & five.
```

```
J=length(Ws.structure)-2; % Corresponds to L in [C,L]=wavedec...
lx=Wx.structure(J+2); % Length of original x, see Wav. Toolb.
ls=Ws.structure(J+2); % Length of original s.
y=zeros(1,lx); % Preallocate output y
```

```
for t=0:lx-ls; % Upper limit on MF output
    for j=find(which_d_scales(2:J+1)) % which levels to include
        [q0, q1]=factor2jb(t,j); % should maybe make a global table
        k=J+1-j;
        % W.cD{q0+1}{j} is to contain cD(J+1-j),qv wavelet t.b
        lcoefs_s=length(Ws.cD{q0+1,k}); %can be optimized

        y(t+1)=y(t+1)+...
            sum( Wx.cD{k}(q1+1:q1+lcoefs_s).* Ws.cD{q0+1,k});
```

```
    end
end
%
%approximation now
%
if which_d_scales(1)==1,
    j=J
```

```
for t=0:lx-ls
    [q0, q1]=factor2jb(t,j); % should maybe make a global table
    lcoefs_s=length(Ws.cA{1+q0}); % 'const' length
    y(t+1)=y(t+1)+sum(Wx.cA(q1+1:q1+lcoefs_s).* Ws.cA{1+q0});
end
```

- Script for testing

```
% dettst.m - script for testing detectors
```



```

% Amund Solvi Bremer, FFI-E
% uncomment the following line to convert the
% script into a function (useful for profiling).
%function dettst();

type='db6'
inputdata='signal'
inputtrans='bank'
load(inputdata)
load(inputtrans);

fs=600; % Sampling frequency

detailbank=[ 0  1 1 0 0 ; 0  0 1 0 0 ]
differentbands=size(detailbank,1);
J=4;
Nfilters=size(bank,2)

Wx=tiwt(x,J,type); %DWT of signal
for i=1:size(bank,2);
Wbank(i)=wt(bank(:,i),J,type); %DWT of transients
end
for i=1:differentbands
details=detailbank(i,:)
y(i)=detectb(x,bank,Wx,Wbank, details);
end

t=[0:length(x)-1]/fs; %time axis

for i=1:differentbands+1
figure(3)
if i==1
subplot(differentbands+1,1,i)
plot(t,x)
title('Singal and matched filter output - time')
else
subplot(differentbands+1,1,i)
plot(t,y(i-1).tdmf)
end
figure(4)
if i==1,plot(t,x)
subplot(differentbands+1,1,i)
title('Signal and matched filter output - transform')
else
subplot(differentbands+1,1,i)
plot(t,y(i-1).wdmf)
title(['bands no ' int2str(detailbank(i,:))])
end
end
end

```

- Auxillary programmes

```
function y=mfcorrelator(x,s)
%correlates x & s, takes central part, pwz.
%06.08.97
ls=length(s);
lx=length(x);
y=filter(s(ls:-1:1),1,x); %This is the fastest way
y(1:ls-1)=[];
y(lx)=0;
```

```
function [q0,q1]=factor2jb(x,j)
% factor2jb.m - factors input x as 2^j*q1 +q0,
% where b<2^j.
% By Emil Urnes, FFI
% mod. A.S. Bremer
q1=floor(x/2^j);
q0=x-q1*2^j;
```

```
function out=qfactor(q0)
% internal functon for tidwt.m.
% Amund Solvi Bremer, FFI-E
if q0 <= 0
    error('q0 zero or negative')
elseif q0 == 1
    out=0;
else
    out=floor(log2(-0.5+q0));
end
```

```
function X = rshift(x,n);
% Shifts x n elements to the right, pads with zeros
% Amund Solvi Bremer, FFI-E
X=zeros(1,length(x)+n);
X(n+1:length(x)+n)=x(1:length(x));
```